THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Remark on definition of topology

Objective: This note aims to disucss a common mistake in Quiz 1 and clarify the definition of topology. Note that in Quiz 1, marks will not be deduced if you have the following mistake.

There are two common definitions of topology. The first one involves three axioms:

Definition 1 *Let X be a set. A subset $\mathfrak{T} \subset \mathfrak{P}(X)$ is called a* topology *of X if it satisfies*

0. *$\emptyset, X \in \mathfrak{T}$;*
1. *Arbitrary union of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$;*
2. *Finite intersection of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$.*

The second one involves only two axioms:

Definition 2 *Let X be a set. A subset $\mathfrak{T} \subset \mathfrak{P}(X)$ is called a* topology *of X if it satisfies*

1. *Arbitrary union of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$;*
2. *Finite intersection of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$.*

Two definitions are actually equivalent to each other. Clearly the first one implies the second one. To show that the second one implies the first one, note that the empty set and the whole space can be constructed by union and intersection as follows:

$$\bigcup_{\text{empty collection}} U = \emptyset \quad , \quad \bigcap_{\text{empty collection}} U = X$$

It is because

- $x \in \bigcup_{\text{empty collection}} U$ if and only if there exists some V inside the empty collection such that $x \in V$. Since there is no set inside the empty collection, the statement must be false.
- $x \in \bigcap_{\text{empty collection}} U$ if and only if for all V inside the empty collection, we have $x \in V$. In other word, $x \notin \bigcap_{\text{empty collection}} U$ if and only if there exists some V inside the empty collection such that $x \notin V$. Since such V cannot be found, the negation is always false and hence the statement is always true.

This shows that two definitions are equivalent. However, it *does not* means that when we check whether a set is a topology, we do not need to check the condition that $\emptyset, X \in \mathfrak{T}$. The correct interpretation should be:

- when we check the condition 'Arbitrary union of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$', we also need to check $\emptyset \in \mathfrak{T}$;
- when we check the condition 'Finite union of elements of $\mathfrak{T}$ lies in $\mathfrak{T}$', we also need to check $X \in \mathfrak{T}$.

For example, consider $X = \mathbb{R}$ and $\mathfrak{T} = \{(a, b) \mid a < b\}$. Note that $\mathfrak{T}$ is *not* a topology since it does not satisfy axiom (1) (for $\emptyset \notin \mathfrak{T}$) and axiom (2) (for $X \notin \mathfrak{T}$).